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Diversification and Capital Gains Taxes with Multiple Risky Assets

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Abstract

We examine the impact of capital gains taxes upon the structure of an investor's optimal portfolio in the presence of multiple risky assets. Our numerical solutions suggest that the diversification benefits of reducing the exposure to a highly volatile concentrated position significantly outweigh the tax costs of selling, even for elderly investors. The presence of multiple risky assets in which the investor earns a substantial risk premium strongly increases the diversification incentive. We also contrast the impact of capital gains taxes and traditional transaction costs on rebalancing decisions and show that it can be optimal for the investor to reduce his overall equity exposure by selling underweighted assets with relatively small capital gains. Finally, we discuss the general qualitative features of the optimal investment policy in a broader context. Both our numerical and qualitative analyses show how the realization decision on one asset depends upon the embedded gains on other assets.

1. Introduction

A central problem confronting investors is how to rebalance their portfolios in the presence of capital gains taxes. Our aim in this work is to examine portfolio rebalancing among *multiple* risky assets. The optimal trade-off between diversification and taxes is a challenging problem because of the endogeneity of the investor's realization decisions and the large dimension of the state space upon which the investor's decision rules are defined. Recently numerical solutions have been obtained and analyzed by Dammon, Spatt and Zhang (2000, 2001), focusing upon a model in which there is a risk-free asset and a *single* risky asset for which capital gains taxes at the time of sale are determined by the weighted average purchase price. These restrictions reduce the dimension of the underlying state space, facilitating numerical solutions. This paper extends the analysis of Dammon, Spatt and Zhang (2001) by analyzing the optimal asset allocation and rebalancing problem in the context of multiple risky assets.[1] For example, we show how an investor with a large concentrated portfolio position (which might arise as a by-product of his employment compensation) should rebalance his asset allocation.[2] More generally, we examine how the investor's alternative diversification opportunities and his tax costs influence his optimal realization behavior.

The phenomenon of an investor being overweighted in risky securities as a consequence of his desire to defer the realization of capital gains often arises as a by-product of the investor's holdings in a particular company. Many successful individuals have their financial net worth strongly linked to the stock price movements of their own firm as a result of their stock and option participation and the strong increase in asset values in the last two decades.[3] While individuals with concentrated holdings

[1] Building upon the modeling strategy in Dammon, Spatt and Zhang (2001), recent papers by Garlappi, Naik and Slive (2001) and Gallmeyer, Kaniel and Tompaidis (2001) also solve numerically portfolio problems with two risky assets in the presence of capital gains taxes. These papers, however, have a slightly different focus than ours. Garlappi, Nail and Slive (2001) focus upon the relationship between capital gains taxes and transaction costs and highlight the nature of the "no-trade" region. Gallmeyer, Kaniel and Tompaidis (2001) focus upon the optimal portfolio decisions when short-sale constraints are relaxed.

[2] Both moral hazard and adverse selection lead firms to provide a significant portion of the compensation offered to key personnel in the form of restricted equity and options, tying much of the employee's compensation and net worth to the success of the firm. Consequently, in practice the compensation contract may constrain the executive's ability to liquidate his concentrated portfolio position, even absent taxes. In fact, Ofek and Yarmack (2000) document that managers are anxious to shed exposure to their firms' equity when they are able to do so.

[3] A static treatment of the risk–return trade-off for concentrated holdings in the presence of taxes is given in Stein, Siegel, Narasimhan and Appeadu (2000).

in a particular firm would like to reduce their holdings to limit their exposure to idiosyncratic (nonpriced) risk, the tax cost of selling positions with low bases looms very large. Of course, financial theory teaches that in the absence of tax costs or incentive constraints, the investor would eliminate his idiosyncratic risk and hold his desired risk exposure through the market portfolio. In a taxed economy the trade-off between such diversification incentives and the desire to defer the capital gains tax (and potentially avoid it at death) is central to the investor's portfolio decision.

To address the problem confronting an investor with a concentrated (overexposed) holding in an asset with substantial idiosyncratic risk, we solve a dynamic setting with a risk-free asset and two risky assets. We set the parameters so that the risky assets have the same expected return, but quite different standard deviations, and set the correlation between the assets to be such that the optimal holding of the higher volatility asset is zero in a tax–free economy. (The standard deviation of the risky asset portfolio is minimized with zero holding of the higher volatility asset by assumption. The lower volatility asset can be interpreted as analogous to the market portfolio and as not possessing idiosyncratic risk, while the higher volatility asset possesses substantial idiosyncratic risk.) Our numerical solutions suggest that introducing capital gains taxes in such a specification leads the investor to realize much of his capital gains on the high volatility asset, provided that the investor's life expectancy is more than several years. This suggests that the diversification effect may dominate the tax disadvantage of selling provided that the investor's horizon is sufficiently long. The horizon (or investor's age) plays a prominent role in this analysis because the diversification advantage of selling (and rediversifying) increases with the investor's horizon, while the tax advantage of the potential reset of the investor's tax basis to the current market value at death becomes more distant. Selling much of the investor's holding of the high volatility asset and potentially substituting the lower volatility asset and leveraging up would allow the investor to enhance his (pre-tax) expected portfolio return without increasing his overall portfolio variability. While Dammon, Spatt and Zhang (2001) suggest in a single risky asset setting that the investor will optimally retain much of his appreciated position to defer the capital gains tax liability, the presence of a substitute risky security through which the investor can earn the pre-tax risk premium greatly enhances the investor's willingness to realize capital gains. This dramatically alters the extent to which investors should incur capital gains taxes to rebalance their portfolios.

Surprisingly, we find that an investor may optimally sell both underweighted as well as overweighted assets in scaling back his overall equity exposure. For example, in our setting in which the investor maximizes intertemporal expected utility we show that to reduce his overall exposure to equity it may be optimal to sell an underweighted asset with a relatively small capital gain in order to reduce the sale of an overweighted asset with a relatively larger capital gain.[4] Obviously, the potential value of reducing the investor's exposure to an underweighted security is sensitive to the relative capital gains tax liabilities. Underweighted and overweighted assets can potentially serve as substitutes for scaling back exposure in maximizing the investor's expected utility objective, unlike in a framework in which the investor is penalized for departures from an exogenous portfolio target (see, for example, Leland (2000)). It can be optimal for the investor to adjust some or all of his assets in solving his portfolio problem, depending upon the relative marginal valuations of the investor's positions.

We also consider an example with a riskless asset and symmetric return distributions for two risky assets. In this situation the optimal portfolio in a tax-free economy (or in a taxable economy with identical tax bases and initial holdings) weights the two assets equally. This allows us to quantify how differences in bases across assets influence the composition of the investor's portfolio. The investor's optimal portfolio is relatively skewed towards the asset with the larger gain and is sensitive to the basis of the alternative risky asset, emphasizing the importance of cross- (substitution) as well as own-basis effects in determining the structure of an investor's optimal portfolio. While for many parameter values the investor adjusts his optimal portfolio holdings for diversification reasons, as in models with transaction costs, the capital gains tax costs triggered by sales lead to

[4] We describe an asset as underweighted (overweighted) if the investor owns relatively less (more) of it compared to an optimal portfolio structure in an economy in which the investor did not face any capital gains tax liabilities because his bases equaled the current market values.

a "no-trade" region in which the investor does not rebalance his portfolio or adjusts only a subset of assets. We examine the dependence of the "no-trade" region upon the investor's age and the basis–price ratio of the underlying assets.

Because of the dimension of the state space, we are unable to obtain *quantitative* solutions for problems with a large number of risky assets. Instead, we derive *robust qualitative* implications for optimal portfolio holdings and trading in the presence of capital gains taxes. Of course, our numerical examples with two risky assets illustrate *quantitatively* how the presence of multiple risky assets influences the investor's trade-off between the benefits of portfolio rebalancing and the tax cost of capital gains realizations.

The qualitative approach we take allows us to extend the framework in Dammon, Spatt and Zhang (2001) to incorporate multiple risky assets and to distinguish the tax basis of different positions in a given stock owned by the investor. Allowing for multiple risky assets and multiple tax bases for each asset enhances the value of the investor's tax-timing option. In addition, the qualitative properties we derive are valid even in settings in which (a) the investor possesses stochastic labor income that is imperfectly correlated with the returns on risky securities and/or (b) asset returns are correlated over time (so that investment decision rules are conditional upon the anticipated distribution of returns). Finally, the conclusions are largely robust to the specification of investor preferences. The qualitative approach allows us to extend many of the intuitions from two–asset models to multiple–asset settings and offers new insights about portfolio effects. For example, one important feature is the *cross–basis* effect that increasing the *distribution* of bases on one asset results in (weakly) reduced sales of the other assets.

Section 2 presents numerical solutions in a simplified framework in which the investor is overexposed to an asset with substantial idiosyncratic risk and also can invest in another risky portfolio (such as the market portfolio) and a risk-free asset. Detailed discussion of the trade-off between the diversification benefit and tax cost is provided. Section 2 also examines a symmetric two risky-asset example to further illustrate the interaction between diversification and taxes and examine the nature of the investor's "no-trade region." In Section 3 we examine broad qualitative insights about the structure of the investor's portfolio choices. Some concluding comments are offered in Section 4. The formal model underlying the numerical solutions is detailed in the Appendix.

2. Numerical Examples — Two Risky Assets

We present numerical results for the optimal investment problem in the presence of capital gains taxes and two risky assets to highlight distinctive findings that arise with multiple risky assets. Our first example illustrates the situation confronting an investor with a highly concentrated portfolio including investment in a volatile individual company stock as well as the market index. We then describe a symmetric example with two risky assets in which there are genuine opportunities for portfolio diversification in the construction of the investor's portfolio, even absent taxes.

The investor makes optimal consumption and portfolio allocation decisions annually starting at age 20 ($t = 0$) and lives for at most another 80 years until age 100 ($T = 80$). The investor's preferences are represented by a constant relative risk averse utility function so that the investor's consumption and portfolio choices are homothetic with respect to his wealth. We assume that the investor derives his income only from financial assets as the presence of labor income would require additional state variable(s) in our formulation unless the investor's labor income was proportional to his wealth. Capital gains are taxed upon sale of the assets. To keep the state space from expanding, the weighted average tax basis is used for both the company stock and the stock index. Our two risky–asset economy then requires four continuous state variables: the basis–price ratios (p_c^* and p_i^*) and initial asset proportions (s_c and s_i) for each of the risky assets. At the investor's death, his tax basis is reset to the current market price.[5] For simplicity we assume that the investor derives utility from his bequest equal to the utility his beneficiary would derive if the bequest were used to purchase an annuity contract offering a constant real consumption stream over H periods, where H measures the strength of the bequest motive. Wash sales are permitted and transaction costs are

[5]Following Dammon, Spatt and Zhang (2001), we assume in our numerical implementation that at death the investor is forced to reset the tax basis on both gains and losses.

zero (so that investors can immediately repurchase shares). The detailed model description, which extends the formulation developed in Dammon, Spatt and Zhang (2001), is given in the Appendix. The annual mortality rate is calibrated to match the life expectancies for the U.S. population.[6]

2.1 Company Stock and Market Index

In the first numerical example, we assume that the investor can invest in three assets: a risk-free bond, a highly volatile risky individual "company" stock, and a risky stock index. We assume that the nominal pre-tax interest rate on the risk-free bond is $r = 6\%$ per year, the nominal dividend yield on both the company stock and the stock index are $d_c = d_i = 2\%$ per year, and the annual inflation rate is $i = 3.5\%$. The nominal capital gain return on the individual company stock is assumed to follow a binomial process with an annual mean and standard deviation of $\bar{g}_c = 7\%$ and $\sigma_c = 40\%$, respectively. The nominal capital gain return on the stock index also is assumed to follow a binomial process with an annual mean and standard deviation of $\bar{g}_i = 7\%$ and $\sigma_i = 20\%$, respectively. These volatilities are reflective of the volatilities of individual stocks and the market portfolio, respectively. The correlation between the capital gain return of the company stock and the stock index is assumed to be $\rho = 0.5$. The parameter values for the risky asset returns are chosen so that the two assets have identical risk premia and a correlation coefficient such that the investor would optimally hold no company stock in the absence of taxes. We assume that the tax rate on dividends and interest is $\tau_d = 36\%$ and the tax rate on capital gains and losses is $\tau_g = 20\%$.[7] The investor is assumed to have power utility with an annual subjective discount factor of $\beta = 0.96$ and a risk aversion parameter of $\gamma = 3.0$. We set $H = \infty$ so that the investor's utility is defined by providing his beneficiary a perpetual consumption stream from the bequest. For numerical tractability, we solve the model using a grid of $(31 \times 31 \times 31 \times 31)$ over the following ranges: $s_c \in [0, 1]$, $p_c^* \in [0.05, 1.05]$, $s_i \in [0, 1]$, and $p_i^* \in [0.05, 1.05]$. Our numerical procedure is

[6] The mortality rates are calculated from the life expectancies in the 1980 Commissioners Standard Ordinary Mortality Tables.

[7] Because of portfolio offset rules we assume that capital gains and losses are taxed identically. We set the tax rate at $\tau_g = 20\%$, because if the investor was overexposed to equity he could rebalance his position by selling some stock and paying tax on his gain at that rate.

summarized at the end of the Appendix.

We begin the discussion of our numerical solutions with the optimal investment policy for the investor at age 99 just before the terminal date. Since the investor dies with certainty next year, this is a one-period optimization problem. Figure 1 shows the optimal overall equity proportion, optimal company equity proportion, and optimal stock index proportion plotted against the basis-price ratios for the company stock and the stock index. The initial company stock and stock index holdings are set at $s_c = 0.9$ and $s_i = 0.1$ (for all three panels) to reflect a situation in which the investor is substantially overexposed to a concentrated equity position and underexposed to the market index. Because the company stock is much riskier than the stock index ($\sigma_c = 2\sigma_i$) and the diversification benefits are insufficient to induce the investor to own the high volatility (company) stock in a tax-free economy, the investor always scales back substantially his holdings of the company stock. Despite the efficiency of bearing risk through the index, the investor does not add to his ownership of the index, when the investor retains more than 25% of his overall assets in the company stock (which corresponds to a gain on the company stock of more than 80%). This is because the investor is still overweighted in equity on an overall basis. As the gain on the company stock and tax cost of selling it decline, the investor sells more of the company stock (and purchases more of the index) to improve the risk–return trade–off of his portfolio. This shows that the diversification benefit is so large that it outweighs the immediate tax costs of trading, even though the tax on any embedded capital gain would be forgiven in the next period (at death).

The panel on the right shows the optimal company stock holding as a function of the tax bases on the two assets. The optimal company stock proportion is approximately zero when the investor has an embedded loss on the company stock (the right back edge of the panel, $p_c^* \geq 1$). This is not surprising because the parameter values were chosen so that the investor optimally holds none of the company stock in his portfolio in the absence of capital gains taxes. Furthermore, since all gains and losses are untaxed at the time of death there is no tax-timing option for investments made at age 99. Although there is no tax-timing option, the investor takes advantage of the tax forgiveness (reset provision) at death by retaining some of his shares of the company stock with very large embedded gains. Because the investor is overweighted in the company stock, he also sells some of his initial company stock holding to reduce his exposure. The extent that the investor liquidates his company stock holding is inversely related to the size of its embedded gain (as in a single risky asset model), but also is increasing in the size of the embedded gain on the stock index.

The last panel shows the stock index proportion as a function of the bases on the two assets. The optimal costless rebalancing stock index proportion is given by the right front edge (the edge corresponding to $p_i^* \geq 1$). The optimal costless rebalancing stock index proportion decreases as the embedded gain on the company stock increases. When the embedded gain on the company stock is large and the embedded gain on the stock index is small, the investor reduces his stock index holding (potentially to zero) to maintain an overall balanced equity proportion and retains more of his company stock holding to minimize his tax costs. When the basis-price ratio on the company stock is less than 0.4 (the corresponding embedded capital gain is greater than 150%), the investor only holds the company stock and not the stock index. This example provides the interesting observation that because of differential tax costs of altering his exposure on different assets due to differences in the basis–price ratios on these assets, an investor overexposed to equity on an overall basis may choose to sell an asset to which he is underexposed, while retaining more of the asset to which he is overexposed. In contrast, in examining a model with a fixed target portfolio and transaction costs (or capital gains taxes) Leland (2000) finds that the investor does not reduce his exposure to an underweighted asset due to the exogenous portfolio targets assumed. In our setting the optimal portfolio structure is endogenously (simultaneously) determined across assets. Note that when the embedded gain on the stock index is large, the investor retains his initial stock index holding. This is illustrated by the flat stock index proportion in much of the figure.

Our discussion above points to the cost of diversification and the reset of the tax basis at death as the reasons for the investor to hold the much riskier company stock. In particular, at age 99 the investor retains much of his initial company stock holding with a large embedded gain to take advantage of tax forgiveness at death. At younger ages, the investor's mortality rate decreases dramatically, making the benefits of tax forgiveness at death relatively small. This should increase the investor's willingness to sell assets that are overweighted. To see how much the tax deferral influences the investor's decision to liquidate the risky company stock, we examine the portfolio decisions at age 90. In Figure 2 we plot the overall equity proportion, the company stock proportion, and the stock index proportion as a function of the tax bases on the two assets. The initial company stock and the stock index holding are set at $s_c = 0.9$ and $s_i = 0.1$, respectively, as in Figure 1.

The overall equity proportion is relatively flat in the basis–price ratio of the company stock and the stock index at age 90. The investor liquidates a large proportion of his initial holdings of the company stock to rebalance his portfolio (he retains only about 10% of his assets in the company stock, even when his embedded gain is very large ($p_c^* = .05$)). Examining the plots for the company stock and the stock index proportions indicates that the decline in the total equity proportion at age 90 is primarily due to the liquidation of the investor's initial company stock holding. The investor is willing to retain a small amount of the concentrated highly appreciated position despite its higher own volatility, because at the margin the investor would prefer to avoid the tax cost, but the nonlinear nature of the diversification costs makes those costs too large for substantial holdings of the company stock.[8] In addition to selling most of his position in the company stock when the capital gains tax cost is small (company stock basis near one), the investor purchases the index since it is the efficient way to bear risk. A cross-basis effect arises since the sale of larger amounts of company stock for higher values of the company stock basis (due to the lower tax cost) results in a larger proportion of the investor's wealth being used to purchase the index. Intuitively, the investor sells much of his exposure to the company stock with an embedded gain in order to use

[8]In the special case in which there is a single risky asset, the concavity of the investor's optimization problem implies that if the investor is greatly overexposed to the risky asset and needs to scale back, his optimal policy is then independent of the investor's incoming holding of the risky asset. This is a consequence of the increasing marginal disutility of being further from the optimal holding of the risky asset. In this region the optimal investment holding is a function of the basis–price ratio and investor's age, but not the incoming exposure. This is analogous to the result in a model with one risky asset and a riskless asset as well as proportional transaction costs that investors with convex disutility for deviations from an exogenous target portfolio adjust the weight on the risky asset to an optimal boundary (e.g., as illustrated by Leland (2000)).

the index to achieve his desired equity exposure.[9]

An interesting contrast at age 90 is offered by Figure 3 in which the investor's initial portfolio is 50% stock index and 50% company stock. In this case the investor retains substantially less company stock than the 10% holding retained above at age 90, when he has large embedded gains on the stock index. This reflects the investor's greater holdings of the index and the tax costs of the index position. For example, if the investor's basis were .05 on both the company stock and the index, then the investor would adjust his holding of the company stock to near zero because to the extent that the investor retains risk exposure he prefers to hold the lower volatility asset (the index). Notice that the lower the investor's basis in the index, the greater the investor's holding of the index and the lower his retention of the company stock, again illustrating the cross–basis effect on asset ownership.

Our example reflects a situation in which the investor is overexposed to a stock in which he would bear considerable idiosyncratic risk if retained. However, it is still striking that even for investors with relatively short horizons (i.e., age 90 in our example) the diversification benefits outweigh the tax cost of selling the high–volatility asset, despite the tax forgiveness on the appreciation if the investor retains that position until his own death.[10] By selling the company stock and using the proceeds along with leverage to reestablish the overall risk of the investor's portfolio, the expected return on the portfolio could be substantially increased. This additional expected return understates substantially the risk-bearing cost because it is not optimal for an investor to leverage up his exposure to the stock index in this fashion. This example suggests the potential value for an investor with a concentrated holding to scale back substantially that exposure to avoid excessive risk bearing, as long as the investor is more than a few years from his anticipated death. In fact,

[9]Note that the figure suggests that the investor will purchase slightly more of the index when his existing gain on the index is small (large index basis). This is a result of the averaging in purchases at unfavorable bases and the wealth effect of smaller embedded tax liabilities. (Dammon, Spatt and Zhang (2001) provides more detailed discussion of this in the context of a single–asset model.)

[10]Of course, in practice investors may have other mechanisms available to reduce the idiosyncratic (nonpriced) exposure of an investor with a large concentrated and highly appreciated position. Wall Street has created a variety of transactional mechanisms to transform the customer's risk and reduce the idiosyncratic risk that the customer needs to bear, while continuing to defer the capital gains taxes.

the incentive to scale back his large exposure to idiosyncratic risk is even stronger at younger ages due to the lower mortality risk and greater diversification benefit.

Our conclusion that the investor should desire to rebalance his portfolio despite capital gains taxes on the appreciation is sensitive to the structure of incremental risk being borne. For example, in the context of a relatively diversified portfolio structure, it would be beneficial for the investor to bear some incremental additional idiosyncratic risk in order to defer capital gain tax realizations. The idea that an investor with a concentrated exposure should be anxious to realize his gain to rediversify as long as the investor has more than a few years of life expectancy seems to contrast sharply with the analysis in Dammon, Spatt and Zhang (2001) with a single risky asset in which an investor with substantial gains may retain much of his equity position in order to continue to defer the implicit capital gains tax liability. However, an important feature of that setting (as a by-product of the assumption of a single risky asset) is the absence of substitute risky assets through which the investor can earn the pre-tax risk premium. In contrast, the presence of alternative risky portfolio opportunities allows the investor to sell appreciated positions to diversify without limiting the investor's opportunity to earn the risk premium, making the sale of appreciated assets much more attractive.

2.2 Diversification: A Symmetric Example

The overall structure of the second example reflects symmetric opportunities across the assets. We assume $\bar{g} = 10\%$ and $\sigma = 30\%$ for each of the two assets. The parameter values and assumptions are otherwise identical with those in the example in Section 2.1, including the assumed correlation of 0.5. In our illustration we will focus on a situation in which the incoming holding of each asset is 0.5. In Figures 4 and 5 we plot the holdings at ages 99 and 40, respectively, given the basis–price ratios on the two assets. Given the assumed parameter values the investor is overexposed to equity when he holds 50% of his portfolio in each of the two risky assets. The only potential asymmetry in the example is due to differential tax bases, allowing us to highlight the effects of capital gains taxes.[11] Of course, the optimal holdings of individual asset equity holdings are identical in the taxable economy (or in a tax-free benchmark economy) for the two assets because of the symmetry of the underlying example. The total equity holdings also are symmetric in the two bases.

The structure of the investor's portfolio at age 99 (Figure 4) illustrates a variety of important effects. In this example the investor is overexposed to each of the two assets (the optimal holding for an asset can be obtained by settings its basis to one given the basis of the other asset) and does not have any incentive to purchase either asset. However, the investor does not necessarily scale back his holding of either asset. In fact, if the bases of both assets are below 0.55, neither asset is sold. There is an interesting cross-basis effect in the example in that if the investor has a small gain on one asset he sells a larger proportion of this asset relative to the asset with the higher gain (however, because of diversification he does hold positive amounts of both assets). There is a large combination of bases for which the investor does not rebalance his portfolio or adjusts only one of the two assets. The shape of the region also exhibits the "cross-basis" effect. Interestingly, the "cross-basis" effect is quite dramatic in that if the investor has no gain on one of the assets and a gain of at least 30% on the second asset, then the investor holds almost his entire equity exposure through the second asset (e.g., as illustrated by the right edge of the figure in the bottom panel). Notice that in this symmetric example in which the investor is overexposed to both assets, there are no basis combinations for which the investor purchases either asset.

Figure 5 illustrates that at age 40 the investor scales back his holdings of both assets for all basis combinations. This reflects the strength of the diversification incentives given the investor's overexposure to equity and remaining horizon. Of course, the tax costs limit the extent of the investor's trading, despite the diversification benefit. The cross (and own basis) effects are quite strong in this situation. For example, at age 40 (Figure 5) if it is costless to sell both assets ($p_c^* \geq 1$ and $p_i^* \geq 1$), then the investor will hold 22.4% of his portfolio in each, while if the basis on one of the assets is .05 (and the other is again one) then the investor will hold 32.8% of his portfolio in the

[11] In the figure we refer to one of the assets as the "index" and the other as the "company" stock (as in Section 2.1), but here the assets are interchangeable.

asset with a basis of .05 and 15% of his portfolio in the asset whose basis is one. This illustrates the considerable sensitivity of investor decisions to taxes and provides an interesting example of how optimal diversification is distorted by taxes. Similar effects arise at other ages. For example, at age 80 (not shown) the investor holds 22% of his portfolio in each of the two assets if their bases are both one, while if the basis on one of the assets is .05 (and the other asset has a basis of one) then the investor will hold 41.2% of his portfolio in the asset with the low basis and 11.8% of his portfolio in the asset whose basis is one.[12] Interestingly, much of the holdings of an asset as its capital gain increases is offset by a reduction in the holding of the other asset.

While Figure 5 is cast in terms of the investor's bases on the two assets, the traditional "no-trade" region in models of transaction costs (e.g., as illustrated by Leland (2000)) is expressed in terms of the investor's holdings of the assets.[13] In Figure 6 we present a counterpart to the traditional "no-trade" region in our example with symmetric returns distributions. For an investor at age 80, Figure 6 provides bounds on the "no-trade" region in terms of the investor's holdings for a given combination of bases (identical basis–price ratios for the assets of .05, .5 and .8 are depicted in the figure). The "no–trade" region is largest for an investor with a low basis–price ratio (large capital gains tax liabilities), since an investor with large transaction costs will have a relatively wide "no-trade" band. In contrast, when the basis–price ratio is high (the capital gains tax liability is small when the basis–price ratio is .8), the "no-trade" interval will be relatively tight as the investor will typically adjust his holdings.[14] The diamond-shaped nature of the region in which the investor

[12] We increased the capital gains tax rate to $\tau_g = 36\%$ (not shown) to examine the sensitivity of the investor's holdings and diversification decisions to his tax rate. At age 80 (not shown) the investor holds 23.9% of his portfolio in each of the two assets if their bases are both one, while if the basis of one of the assets is .05 (and other asset has a basis of one) then the investor will hold 50% of his portfolio in the asset with the low basis and 11.5% of his portfolio in the asset whose basis is one. Not surprisingly, there is greater sensitivity of the investor's holdings to each asset's own basis as well as the cross basis of the other asset when higher capital gains taxes make sale of an appreciated position and diversification more costly.

[13] For each asset, the "no-trade" region is defined as the region for which the incoming stock holding and the outgoing stock holding are the same as a fraction of beginning-of-period wealth. For each asset, the "no-trade" region is defined by two boundaries. If the incoming stock holding is below the lower boundary or above the upper boundary, the investor will adjust his holding of this asset. Otherwise, he will retain his current holding. For the two-dimensional plot, the two relatively steep lines show the lower and upper bounds for the company stock (the first stock) and the two relatively flat lines show the lower and upper bounds for the stock index (the second stock).

[14] The size of the "no-trade" region is slightly narrower for an investor at age 40 (not shown) than at age 80, since the incentive to adjust one's portfolio holdings at age 40 will be somewhat greater.

does not alter his holding of either asset (this shape also arises in Leland's (2000) "no-trade" region) reflects the cross-basis effect in the optimal portfolio structure.

3. General Qualitative Features

We now analyze the impact of capital gains taxes upon the structure of an investor's optimal portfolio with multiple risky assets from a broader perspective. In this section we examine general qualitative features of capital gains realization behavior and portfolio choice in the presence of capital gains taxes, extending the framework in Section 2. The investor can allocate his financial wealth among J risky assets (rather than two risky assets) and a risk-free asset. We also now permit (though do not require) the capital gains tax liability to be determined by the specific tax basis of the positions sold, rather than requiring the tax liability to be calculated using the investor's average basis. In addition, unlike Dammon, Spatt and Zhang (2001) and the framework in Section 2, we allow labor income to be stochastic and imperfectly correlated with asset returns and the investor's wealth, allow asset returns and labor income to be correlated over time, and do not restrict the investor to possess constant relative risk averse preferences (of course, the preferences must satisfy non-satiation and concavity conditions).

As in Dammon, Spatt and Zhang (2000, 2001) and Section 2 of this paper we assume that wash sales are permitted (so that investors can realize tax losses and immediately repurchase their desired exposure) and transaction costs are zero (so that investors can rebalance their portfolio without cost absent capital gains taxes).[15] We assume that tax rates are constant over time and that taxation of capital gains (or losses) occurs at the time of sale. In addition, we assume that the capital gains tax rate is not influenced by the investor's holding period for the asset.

In light of these assumptions, the investor's portfolio choice problem can be formalized as a

[15]The analysis in this section is robust to the presence of tax–deferred (retirement) investing. Dammon, Spatt and Zhang (2000) point out that optimal investment of retirement wealth (and taxable wealth) is not influenced directly by the incoming composition of retirement assets because the investor can costlessly adjust his portfolio composition in the tax–deferred account given the tax treatment of the account and the assumed absence of transaction costs. Under the additional assumption of constant relative risk-averse preferences, only the proportion of the investor's wealth that is tax-deferred and the investor's relative holdings of taxable assets and the extent of any taxable gains and losses influence the optimal relative holdings of the investor's wealth.

dynamic programming problem in which the investor's dollar ownership and corresponding basis (or basis-price ratio) for each distinct lot of shares are the principal state variables. In addition, the investor's age and variables relevant to predicting the distribution of future asset returns and labor income also are state variables.[16]

In this general framework a number of interesting qualitative conclusions emerge.[17] While some of these properties are relatively obvious, others are more subtle.

Property 1 *The investor optimally realizes all losses.*[18]

On any position in which the investor's tax basis is above the current market price, the investor will realize the loss and repurchase shares to reestablish his optimal holdings. Since by assumption there are no wash sale restrictions or transaction costs in our framework, the investor can realize all available losses (speeding up their recognition for tax purposes) without constraining his desired exposure to the specific asset.[19]

The investor would lose the time value on the tax reduction and could lose the opportunity to take the loss by delaying the realization. While we permit the investor to have distinct per-share basis values for different holdings of the asset (in which case the investor would sell those lots with a loss), the result also would apply to situations in which the investor is unable to distinguish the per-share tax bases of his holdings. Not only is it optimal for the investor to realize all his losses, it also may be optimal for the investor to realize small capital gains if the investor desires to reduce his exposure to equity (e.g., for portfolio rebalancing).[20]

[16]The specific state space depends upon the precise model specification. For example, under the assumptions of labor income being proportional to wealth and constant relative risk averse preferences the investor's wealth is not a state variable and in fact, important choice variables are the proportions of investor wealth allocated to the various assets. Variables for predicting the future distribution of asset returns are not relevant when returns are identically and independently distributed over time. Under the assumption that the investor's tax liability depends only upon his average basis, the portfolio decision rules do not need to be conditioned upon the full distribution of investor bases for each asset.

[17]The existence of a solution to the optimization problem follows from continuity of the investor's objective function and the compactness of the choice set (which is satisfied as long as the investor's potential borrowing is bounded).

[18]In various contexts this observation has been made by a number of authors including Constantinides and Scholes (1980), Constantinides (1983), and Dybvig and Koo (1996), among others.

[19]Note that Dammon and Spatt (1996) show that it need not be optimal for an investor to realize losses prior to the end of the short-term region, if there is asymmetric treatment of short-term and long-term realizations.

[20]The investor's optimal portfolio policy is independent of the specific composition of his losses (or gains that it

In solving the portfolio optimization problem in the presence of capital gains taxes, it is helpful to cast the portfolio adjustments in terms of shadow prices. In scaling back the exposure to equity, one should first scale back the asset whose position has the smallest marginal value if retained (lowest shadow price), taking into account the capital gains tax cost. Of course, the shadow prices change as we alter the positions. In particular, as we reduce a position its shadow price (marginal value) increases (reflecting the concavity of the portfolio optimization problem) and the shadow price of other assets declines. In a portfolio context it may be optimal to adjust the holdings of only a single asset if there is enough difference in the shadow prices among the assets.[21] We only would begin to scale back a second asset when the shadow prices of the assets with the two largest shadow prices are equated. The same logic can be extended to additional assets.[22],[23] Of course, these adjustments are not really sequential, but a heuristic/intuitive interpretation.

One simple illustration of the shadow price principle arises when some risky assets are identically distributed (same mean, variance, common cross-correlation) and the investor is overexposed to these assets. If there is an identical gain on these assets in an average basis setting (i.e., the investor's average basis determines the tax consequences of a realization), then the investor sells his largest position until it reaches the size of the next largest position, etc. Of course, if the larger

is optimal to realize) given the after–tax proceeds of his realizations and the remaining locked-in positions of the investor. For example, since it is costless to adjust a position whose basis–price ratio is one, the investor's optimal portfolio is independent of his incoming holdings of such positions. As the examples in Section 2 illustrate, the investor's optimal portfolio proportions are not influenced by the basis–price ratio of an asset in the region in which the basis–price ratio exceeds one.

[21]In a context with an exogenous portfolio target, Leland (2000) demonstrates that it often is optimal to adjust the holdings of only a single asset.

[22]The shadow price principle can be applied to other settings with multiple assets and taxes. For example, in a departure from our framework consider an investor who has made a charitable commitment. Assuming identical return distributions the investor would donate the asset with a relatively large overexposure or relatively large gain. As long as the donation is small relative to the donor's wealth, it is optimal to donate shares of just a single asset. Only for larger donations would donating multiple assets be optimal because of changing shadow prices. As an aside we observe that the charitable donation issue is somewhat different than the rebalancing issue, because capital gains taxes are not paid in the donation case. Therefore, the investor donates low basis shares (this does not depend upon the assumption about future tax rates as the investor always prefers to retain high basis shares), whereas he sells high basis shares to rebalance his portfolio. Consequently, the relevant notion of shadow price in a donation context is different than in the basic diversification problem with capital gains taxes.

[23]In problems with trading frictions such as capital gains taxes, the frictions may lead to a wedge that makes it infeasible to equate shadow prices and instead a corner solution results. This is illustrated by the flat regions for the ownership of the assets (corresponding to retaining the incoming position or corresponding to the short-sale constraint being binding) in the examples in Section 2.

holding had a relatively smaller gain, then the investor would keep selling it until that holding's size was substantially below the other asset's (to equate the shadow prices). If the larger position had the larger gain, then it would be ambiguous which asset to sell first.

Property 2 (High–Basis, First–Out): *The investor will optimally realize positions if and only if the bases for these positions are at least equal to a critical value b^*, where b^* varies by asset and depends upon the current vector of state variables and the parameters describing the evolution of the economy.*[24]

For any future realized path of asset returns and corresponding vector of the number of shares sold today and at the respective future nodes in the tree, the investor's utility will be at least as large if the investor selects the lots to sell in order to maximize the losses (minimizes the gains) realized at the present time. If the investor realized a position with a lower basis (and larger immediate tax gain), while retaining one with a higher basis (and smaller immediate tax gain), the incremental future benefits (including time value cost) of retaining the higher basis position could not exceed that of immediately realizing the higher (rather than lower) basis position.

One way to interpret the conclusion in Property 2 is in terms of shadow prices. If we allow distinct shadow prices for each distinct lot of a given asset, the highest basis (lowest gain) positions, which have the lowest shadow prices, always would be the most valuable to sell. This suggests a complementary intuition for High–Basis, First–Out.

The High–Basis, First–Out characterization implies that only a single decision variable and shadow price is needed for each asset in the investor's portfolio optimization problem.[25] This suggests a sense in which multiple tax bases for a given asset represent a less fundamental feature

[24] This observation was noted in various contexts previously by Balcer and Judd (1987) and Dybvig and Koo (1996), among others.

[25] Of course, high–basis, first–out (or even realizing losses as soon as they become available) may not be an optimal decision rule if the tax rate is anticipated to rise. As a result, the constant tax rate assumption is very useful in limiting the number of decision variables (e.g., to one per risky asset) as well as preventing the investor from realizing gains (at relatively low marginal rates) for the purpose of raising his tax basis (somewhat akin to the argument in Constantinides (1984) and Dammon and Spatt (1996) that investors may find it optimal to realize long-term gains in order to create the option to be able to realize future losses short-term).

of the portfolio optimization with taxes than multiple assets (each using the average basis for that asset). In the presence of multiple bases for a single asset, unlike multiple assets, the ranking of which positions are the most valuable to realize (or defer) is not altered by shocks to asset values.[26]

Corollary 1 *High–Basis, First–Out implies that it is never optimal to sell positions with an embedded capital gain and simultaneously repurchase shares in the same stock at the current market price.*

Selling shares with capital gains while simultaneously repurchasing shares would violate the argument underlying the high–basis, first–out condition. The investor would be better off deferring the realization of the gain on those shares that he repurchases.

Property 3 *The investor's value function is increasing as the distribution of investor tax bases shifts upward (in the sense of "First-Order Stochastic Dominance") for each asset, holding constant the shares of each asset owned by the investor.*

If an investor with relatively higher bases in an asset replicates the optimal decisions of an investor with uniformly lower bases in an asset, the investor with the relatively higher bases would be strictly better off than the investor with lower bases (and can do even better by making the optimal decisions, given his actual bases).[27]

Note that the argument applies even if the distribution of bases for various assets shifts simultaneously in the sense of first-order stochastic dominance. Of course, increasing the investor's tax bases makes the investor effectively wealthier. Since the investor with a higher distribution of bases is effectively wealthier, the investor will consume more than an investor with a lower distribution of bases.[28]

[26] Multiple bases can be represented as a special case of the multiple asset setting with average bases by treating distinct bases for a given holding as distinct assets (whose prices are perfectly correlated, but whose bases differ).

[27] Of course, this same idea applies in a setting in which the investor's "average" basis is the relevant basis for tax calculations, as in the setting in Section 2 or Dammon, Spatt and Zhang (2001). Consequently, in a single-basis setting the investor's value function increases in the basis.

[28] As an aside, note that the consequences of the investor's tax bases for his marginal utility of consumption and asset pricing models are typically not emphasized in finance. One exception to this is Bossaerts and Dammon (1994).

Property 4 *Increasing the distribution of investor tax bases in a specific asset (in the sense of "First-Order Stochastic Dominance") and fixing the investor's bases and holdings in other assets will cause the investor to retain (weakly) less (and sell more) of the asset.*[29]

Increasing the basis distribution reduces the distribution of embedded capital gains and lowers the current cost of selling a fixed amount of appreciated stock. The marginal condition characterizing the optimal amount of the asset that the investor sells (which equates the marginal diversification benefits and tax costs) implies that the investor sells at least as much of the asset (due to the lower marginal tax costs of selling an identical number of shares) and retains no more of the asset after the distribution of bases is increased. Shifting upward the distribution of tax bases (and shifting downward the distribution of embedded capital gains) decreases the shadow price of the investor's holding of the asset so that the investor will sell (weakly) more of the asset and retain (weakly) less of it.

The application of Property 4 to a situation with multiple assets is subtle (unlike Property 3). Property 4 does apply to shifting the distribution of bases for a single asset, including shifting the distribution of bases for one asset in a setting in which there are many other assets. However, suppose that we increase simultaneously the distribution of bases in several assets. We can no longer conclude then that the investor will retain less (sell more) of all the assets subject to these first-order dominance shifts in the distribution of bases. The reason is that in addition to the direct impact of the shift in the distribution of bases upon the holdings of that asset, the shift of the distributions of bases of other assets also influences the sales of the initial asset. In fact, our analysis below suggests that such "cross" effects will work in the opposite direction because of the substitution decisions by investors owning multiple assets.

An interesting issue is how do larger gains on one asset influence the optimal holdings of the other assets? There is no direct counterpart of this issue in option pricing theory since the issue

[29]The result examines the impact of the investor's basis upon his selling decision (i.e., when he is overexposed), but not his purchase decision (i.e., when he is underexposed). In fact, in the case of the average basis rule footnote 9 illustrates a situation in which the purchase decision when the investor is underweighted in the asset actually increases in its basis.

concerns the impact of a change in the exercise price of one asset upon exercise decisions of other assets. Our conclusion focuses upon the idea that assets are substitutes due to the diversification effects and the wealth effect of changes in the distribution of gains.

Property 5 *Fixing the incoming asset allocation and increasing the distribution of bases on one of the assets (in the sense of "First-Order Stochastic Dominance"), the investor will retain (weakly) more (and sell less) of the other assets (both the other risky assets and the risk-free asset).*

By Property 4 the investor sells more of the asset with smaller gains (weakly). The other risky assets are a substitute for diversification reasons. In addition to the substitution effect, there also is a wealth effect induced from the impact of smaller gains on the original risky asset (see Property 3). These tax impacts of the smaller gains make the investor effectively wealthier, which would increase the demand for the other risky assets and the risk-free asset. Consequently, the overall sales of the other risky assets and the risk-free asset will decrease. The examples in Section 2 illustrate that increasing the gain (decreasing the basis) on a risky asset to which the investor is overexposed will cause the investor to retain (weakly) more of that position and (weakly) less of other risky assets to which the investor is overexposed and the riskless asset.

An important related issue is the cross effect of the impact of increases in the incoming holding of an asset upon the holdings of other assets. As the size of the holding of a risky asset with embedded capital gains increases (with a corresponding reduction in the holdings of the risk-free asset), the investor will possess more of an incentive to scale back the other risky assets. This again is a consequence of the assets serving as substitutes due to portfolio diversification.

An interesting issue is the nature of the bias in our framework when we restrict attention to a limited set of assets or bases. To explore the issue we use traditional option pricing arguments to consider the impact of spreading the distribution of bases, while fixing the number of shares owned by the investor.

Property 6 *If the distribution of gains is made more dispersed in the sense of "Second-Order*

Stochastic Dominance," the investor will have a (weakly) larger value function and will be at least as well off.

An investor with a more dispersed distribution of bases can mimic the optimal decisions of an investor with the more concentrated distribution (the argument is analogous to that for our "High–Basis, First–Out" conclusion in Property 2), but has the advantage of additional flexibility by being able to realize only the positions with the largest bases (smallest gains).

Notice that the preference for a more dispersed distribution of bases is robust to the investor being risk averse because the argument relies upon replication. A special case of the second-order stochastic dominance shift is the comparison between the full distribution of bases and the average basis.

Corollary 2 *The investor who uses the full distribution of bases (Specific Share Identification) possesses a more valuable option than an investor using his average purchase price as his tax basis.*

This result is in a similar vein to a portfolio of options being at least as valuable as an option on the corresponding portfolio (e.g., Merton, 1973). Consequently, the value function is greater when there are distinct options on different bases rather than a single option on the overall basket. In particular, it may be optimal to exercise the option on some positions (i.e., those with the highest bases, as in Property 2) and not others.

Because the value of the tax-timing options are understated by the average basis method (used to simplify the problem numerically by Dammon, Spatt and Zhang, 2001) the demand for equity would typically be higher (using the full distribution of bases) than the average basis method would suggest. This would be particularly strong at old ages because the tax-related advantage of equity is then largest (due to the reset provision at death). For example, while the Dammon, Spatt and Zhang (2001) model predicts that the elderly would not be adding additional equity exposure once they had substantial gains (because when using the average price rule, new shares would not provide much ability to benefit from realizing tax losses), in practice elderly investors would have a tax-related incentive to add exposure whose basis equaled the current price to exploit the ability to realize losses while potentially bequeathing appreciated assets without payment of capital gains taxes. This incentive would gradually increase with the investor's age and mortality risk (as illustrated when the investor's average basis equals the current market value by the numerical solutions in Dammon, Spatt and Zhang, 2001).

Investor decisions are much more sensitive to the basis–price ratio for those tax realization options that are "near-the-money" (basis–price ratio close to one). When the basis–price is small there is already considerable lock-in and little remaining sensitivity of investor decisions as there is not much of an option to generate losses.

Analogous to the discussion concerning multiple bases (Property 6), an investor who can invest in individual assets rather than a market index has a larger value function and is better off. Considering the tax realization options, the demand for risky assets in the presence of multiple assets should be relatively larger than suggested by a single risky asset model. The bias may be particularly large for elderly investors because of the value associated with the ability to increase the tax basis on shares left to his heirs and the investor's ability to always realize losses during his lifetime, as well as the larger ability to exploit this option if the investor acquires a diverse set of underlying assets. Together with our argument about diverse bases, this suggests that the single asset and basis framework may understate the demand by the elderly for acquiring risky assets to benefit from the reset provision.

4. Conclusions

An important determinant of the investor's liquidation policy and optimal portfolio holding is the shadow price (the marginal value) associated with the positions currently held. If the investor is overweighted in equity, he should always first scale back the position of risky stock(s) with the smallest marginal value if retained. The idea that the investor adjusts his portfolio based on the shadow price of the positions held is reflected in our first numerical example with two risky assets: a highly volatile risky company stock and a less volatile stock index. When overexposed to equity, the investor first liquidates the volatile company stock because the investor derives a high marginal diversification benefit. This finding has an important implication for investors with a concentrated portfolio holding (such as company executives who receive a significant amount of company stock as compensation). It may be beneficial for these investors to reduce substantially their company stock holdings when permitted, even though sales of their holdings may entail sizable tax costs. However, when the embedded gain on the company stock is very large and the benefit of the reset provision is imminent, the investor reduces the sale of the company stock and sells the stock index instead when the gain on the index is small. This is because the marginal value of retaining the company stock with a large embedded gain is higher than the marginal value of retaining the stock index with a relatively small gain when the investor's life expectancy is short and the investor will benefit from the reset of the tax basis at death. Analogously, in our numerical example with symmetric return distributions and holdings the investor scales back substantially more his holdings of the asset with smaller capital gains, thereby equating the shadow prices of his marginal holdings across assets. Our numerical and qualitative results illustrate how the investor's bases among assets interact in the optimal portfolio structure. The larger the investor's gains on an asset, the greater the investor's sales of substitute risky securities.

References

Balcer, Y. and K. Judd, 1987, "Effects of Capital Gains Taxation on Life-Cycle Investment and Portfolio Management," *Journal of Finance* 42, 743-761.

Bossaerts, P. and R. Dammon, 1994, "Tax-Induced Intertemporal Restrictions on Security Returns," *Journal of Finance* 49, 1347-1371.

Constantinides, G., 1983, "Capital Market Equilibrium with Personal Tax," *Econometrica* 51, 611-636.

Constantinides, G., 1984, "Optimal Stock Trading with Personal Taxes: Implications for Prices and the Abnormal January Returns," *Journal of Financial Economics* 13, 65-89.

Constantinides, G. and M. Scholes, 1980, "Optimal Liquidation of Assets in the Presence of Personal Taxes: Implications for Asset Pricing," *Journal of Finance* 35, 439-452.

Dammon, R. and C. Spatt, 1996, "The Optimal Trading and Pricing of Securities with Asymmetric Capital Gains Taxes and Transaction Costs," *Review of Financial Studies* 9, 921-952.

Dammon, R., C. Spatt and H. Zhang, 2001, "Optimal Consumption and Investment with Capital Gains Taxes," *Review of Financial Studies* 14, 583-616.

Dammon, R., C. Spatt and H. Zhang, 2000, "Optimal Asset Location and Allocation with Taxable and Tax-Deferred Investing," working paper, Carnegie Mellon University and University of North Carolina.

Dybvig, P. and H. K. Koo, 1996, "Investment with Taxes," working paper, Washington University at Saint Louis.

Gallmeyer, M., R. Kaniel and S. Tompaidis, 2001, "Two Stock Portfolio Choice with Capital Gain Taxes and Short Sales," working paper, University of Texas at Austin.

Garlappi, L., V. Naik and J. Slive, 2001, "Portfolio Selection with Multiple Assets and Capital Gains Taxes," working paper, University of British Columbia.

Leland, H., 2000, "Optimal Portfolio Implementation with Transaction Costs and Capital Gains Taxes," working paper, University of California at Berkeley.

Merton, R., 1973, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4, 141-183.

Ofek, E. and D. Yermack, 2000, "Taking Stock: Equity-Based Compensation and the Evolution of Managerial Ownership," *Journal of Finance* 55, 1367-1384.

Stein, D., A. Siegel, P. Narasimhan and C. Appeadu, 2000, "Diversification in the Presence of Taxes," *Journal of Portfolio Management*, Fall, 61-71.

Appendix. The Economic Model Underlying Numerical Examples

The specification of the model is a multiple risky asset version of the framework in Dammon, Spatt and Zhang (2001). The economy consists of investors living for at most T periods, where T is a positive integer. This allows us to directly consider the impact of the investor's age (and increasing mortality) upon his optimal consumption, investment, and realization behavior. Let λ_k be the single-period hazard rate for period k. We assume that $\lambda_k > 0$ for all k and that $\lambda_T = \infty$. The probability that an individual investor lives through period t ($t \leq T$) is given by the following survival function:

$$F(t) = exp(-\sum_{k=0}^{t} \lambda_k) \tag{1}$$

where $0 < F(t) < 1$ for all $0 \leq t < T$, and $F(T) = 0$.

Investors in the economy derive utility from consuming a single consumption good. For simplicity, we assume that all income for consumption is derived from financial assets. Investors can trade $(J+1)$ assets in the financial markets: a riskless one-period bond and J risky stocks. The pre-tax nominal return on the riskless bond is denoted r and is assumed to be constant over time. The nominal payoff to holding one share of stock j from date $t-1$ to date t is $(1+d_j)P_{jt},\ j = 1, \cdots, J$, where d_j is a constant pre-tax dividend yield and P_{jt} is the nominal stock price at date t. We assume that the pre-tax nominal capital gain returns on the stocks are serially independent and follow exogenous binomial processes with a covariance matrix of Σ. No transaction costs are incurred for trading assets. We denote by n_{jt} the number of shares of stock j held *after* trading at time t, and assume that short sales are not allowed ($n_{jt} \geq 0,\ j = 1, \cdots, J$). Nominal dividend and interest payments are taxed at a constant rate of τ_d.

The tax treatment of capital gains and losses is as follows. Any *realized* capital gains are subject to a constant capital gains tax rate of τ_g, while *realized* capital losses are credited at the same rate. To calculate an investor's nominal capital gain, we assume that the tax basis for shares of stocks currently held is the weighted average purchase price of those shares. Denote by P^*_{jt} the nominal tax basis on stock j after trading at time t. The nominal tax basis follows the law of motion:

$$P_{jt}^* = \begin{cases} \frac{n_{jt-1}P_{jt-1}^* + \max(n_{jt} - n_{jt-1}, 0)P_{jt}}{n_{jt-1} + \max(n_{jt} - n_{jt-1}, 0)}, & \text{if} \quad P_{jt-1}^* < P_{jt} \\ P_{jt}, & \text{if} \quad P_{jt-1}^* \geq P_{jt}. \end{cases} \tag{2}$$

The above specification indicates that the updating rule for the investor's tax basis depends upon whether there is an embedded capital gain or loss on the shares currently held, and whether the investor buys or sells shares in period t. In the case of an embedded capital gain (i.e., $P_{jt-1}^* < P_{jt}$), the investor's tax basis is unchanged from the previous period (i.e., $P_{jt}^* = P_{jt-1}^*$) if the investor sells shares in period t (i.e., $n_{jt} < n_{jt-1}$). However, if the investor buys shares in period t (i.e., $n_{jt} > n_{jt-1}$), then the investor's tax basis is equal to a weighted average of the previous tax basis and the purchase price of the new shares, with the weights determined by the number of old and new shares. In the case of an embedded capital loss (i.e., $P_{jt-1}^* > P_{jt}$), the investor's tax basis after trading at date t is equal to the current stock price, P_{jt}. This is due to the fact that, without transaction costs or wash sale rules, it is optimal for the investor to liquidate all shares for tax purposes before rebalancing his portfolio. Hence, any shares held after trading at date t will have been bought at the current stock price.

The investor's problem is to maximize his discounted expected utility of lifetime consumption, given his initial endowment and asset holdings, subject to the intertemporal budget constraint. Since at any given time t an investor has a positive probability of death, the treatment of his terminal wealth is an issue. In this model, we assume that at the time of death the investor's asset holdings are liquidated without payment of the capital gains tax and the proceeds are used to purchase an H-period annuity for the benefit of the investor's beneficiary. This forgiveness of the capital gains tax at death is consistent with the *reset provision* of the current U.S. tax code. We assume that the H-period annuity provides the investor's beneficiary with nominal consumption of

$$\frac{r^*(1+r^*)^H}{(1+r^*)^H - 1} W_t (1+i)^{k-t} \equiv A_H W_t (1+i)^{k-t}$$

at date $k, t+1 \leq k \leq t+H$, where i is the constant rate of inflation, $r^* = [(1-\tau_d)r - i]/(1+i)$ is the after-tax real bond return, W_t is the investor's wealth at the time of death, and $A_H = [r^*(1+r^*)^H]/[(1+r^*)^H - 1]$ is the H–period annuity factor. We assume that the investor and his beneficiary have identical preferences and that the utility derived by the investor from his bequest is equal to the utility derived by the beneficiary. This specification allows us to examine the sensitivity of the investor's optimal consumption and investment policies to the number of periods that the investor wishes to provide consumption support to his beneficiary, with higher values for H indicating a stronger bequest motive.

The investor's problem can now be represented as follows:

$$\max_{C_t, B_t, n_{1t}, \cdots, n_{Jt}} E\left\{\sum_{t=0}^{T} \beta^t \left[F(t)u\left(\frac{C_t}{(1+i)^t}\right) + [F(t-1) - F(t)] \sum_{k=t+1}^{t+H} \beta^{k-t} u\left(\frac{A_H W_t}{(1+i)^t}\right)\right]\right\} \tag{3}$$

s.t.

$$W_t = \sum_{j=1}^{J} n_{jt-1}[1 + (1-\tau_d)d_j]P_{jt} + B_{t-1}[1 + (1-\tau_d)r], \quad t = 0, \cdots, T, \tag{4}$$

$$C_t = W_t - \tau_g \sum_{j=1}^{J} G_{jt} - \sum_{j=1}^{J} n_{jt} P_{jt} - B_t, \quad t = 0, \cdots, T-1, \tag{5}$$

$$n_{jt} \geq 0, \quad j = 1, \cdots, J, \text{ and } t = 0, \cdots, T-1, \tag{6}$$

$$n_{jT} = 0, \; j = 1, \cdots, J, \text{ and } B_T = 0, \tag{7}$$

given the initial bond and stock holdings, B_{-1} and $n_{j,-1}, j = 1, \cdots, J$, and the initial tax basis, $P^*_{j,-1}, j = 1, \cdots, J$. In Eq. (3), $F(-1)$ is set equal to one to indicate that the investor has survived up to period 0, $u(\cdot)$ denotes the investor's utility function, C_t is the investor's nominal consumption at date t, B_t is his nominal investment in bonds at date t, and β is the subjective discount factor for utility. The expression inside the square brackets in Eq. (3) is the investor's probability weighted utility at date t. The first term measures the investor's utility of consumption in period t weighted by the probability of living through period t, while the second term is the investor's utility of his bequest weighted by the probability of dying in period t.

Eq. (4) defines the investor's beginning-of-period wealth, W_t, as the value of the investor's portfolio holdings *before* trading at date t, including the after-tax interest and dividend income but prior to capital gains taxes. Eq. (5) is the investor's time-t budget constraint, where G_{jt} is the *realized* nominal capital gain (or loss) at date t on stock j and is given by:

$$G_{jt} = \{I(P^*_{jt-1} > P_{jt})n_{jt-1} + [1 - I(P^*_{jt-1} > P_{jt})]\max(n_{jt-1} - n_{jt}, 0)\}(P_{jt} - P^*_{jt-1}), \tag{8}$$

where $I(P^*_{jt-1} > P_{jt})$ is an indicator function that takes the value of one if there is an embedded capital loss (i.e., $P^*_{jt-1} > P_{jt}$) and zero otherwise. This formulation exploits the fact that the investor optimally sells all shares with an embedded capital loss to benefit from the tax rebate in the absence of transaction costs and wash sale rules. Hence, with an embedded capital loss on the n_{jt-1} shares held coming into period t, $I(P^*_{jt-1} > P_{jt}) = 1$ and $G_{jt} = n_{jt-1}(P_{jt} - P^*_{jt-1}) < 0$. Since there are no wash sale rules, the sale of stock with an embedded capital loss does not prevent the investor from repurchasing stock in period t ($n_{jt} > 0$) to rebalance his portfolio. With an embedded capital gain, $I(P^*_{jt-1} > P_{jt}) = 0$ and the investor pays a capital gains tax only on those shares that are actually sold in period t. In this case, the total realized capital gain in period t is $G_{jt} = \max(n_{jt-1} - n_{jt}, 0)(P_{jt} - P^*_{jt-1}) \geq 0$, where $\max(n_{jt-1} - n_{jt}, 0)$ is the number of shares sold in period t.

We assume that agents' preferences can be expressed as follows:

$$u\left(\frac{C_t}{(1+i)^t}\right) = \frac{\left(\frac{C_t}{(1+i)^t}\right)^{1-\gamma}}{1-\gamma} \tag{9}$$

where γ is the relative risk aversion coefficient. Note that the summation appearing in the second term of the objective function can be rewritten as follows:

$$\sum_{k=t+1}^{t+H} \beta^{k-t} u\left(\frac{A_H W_t}{(1+i)^t}\right) = \frac{\beta(1-\beta^H)\left(\frac{A_H W_t}{(1+i)^t}\right)^{1-\gamma}}{(1-\beta)(1-\gamma)}.$$

Letting X_t denote the vector of state variables at date t, we can write the Bellman equation for the above maximization problem as follows:

$$V_t(X_t) = \max_{C_t, B_t, n_{1t}, \cdots, n_{Jt}} \left\{ \frac{e^{-\lambda_t}[\frac{C_t}{(1+i)^t}]^{1-\gamma}}{1-\gamma} + \frac{(1-e^{-\lambda_t})\beta(1-\beta^H)\left(\frac{A_H W_t}{(1+i)^t}\right)^{1-\gamma}}{(1-\beta)(1-\gamma)} + e^{-\lambda_t}\beta E_t[V_{t+1}(X_{t+1})] \right\} \tag{10}$$

for $t = 0, \cdots, T-1$, subject to Equations (2) and (4)-(8). The sufficient state variables for the investor's problem at date t consists of the stock price at date t, the tax basis before trading at date t, the stock holdings before trading at date t, and the total wealth before trading at date t. We represent the vector of state variables as follows:

$$X_t = [P_{1t}, P^*_{1t-1}, n_{1t-1}, \cdots, P_{Jt}, P^*_{Jt-1}, n_{Jt-1}, W_t]'. \tag{11}$$

The above problem can be simplified by using beginning-of-period wealth, W_t, as the numeraire. Let $s_{jt} = n_{jt-1}P_{jt}/W_t$ be the fraction of beginning-of-period wealth invested in stock j *prior* to trading in period t, $f_{jt} = n_{jt}P_{jt}/W_t$ be the fraction of beginning-of-period wealth allocated to stock j *after* trading in period t, $b_t = B_t/W_t$ be the fraction of beginning-of-period wealth allocated to bonds *after* trading in period t, $p^*_{jt-1} = P^*_{jt-1}/P_{jt}$ be the investor's basis–price ratio on stock j applicable to trading in period t, $g_{jt} = P_{jt}/P_{jt-1} - 1$ be the pre-tax nominal capital gain return on stock j from period $t-1$ to period t, and

$$R_{t+1} = \frac{\sum_{j=1}^J f_{jt}[1+(1-\tau_d)d_j](1+g_{jt+1}) + [1+(1-\tau_d)r]b_t}{\sum_{j=1}^J f_{jt} + b_t}$$

be the gross nominal return on the investor's portfolio from period t to period $t+1$ after payment of the tax on dividends and interest but prior to the payment of capital gains taxes. Using this notation, Eq. (4) can be written as a linear dynamic wealth equation:

$$W_{t+1} = R_{t+1}(\sum_{j=1}^J f_{jt} + b_t)W_t. \tag{12}$$

Similarly, the budget constraint in Eq. (5) can be written as follows:

$$c_t = 1 - \tau_g \sum_{j=1}^{J} \delta_{jt} - \sum_{j=1}^{J} f_{jt} - b_t \tag{13}$$

where $c_t = C_t/W_t$ is the consumption–wealth ratio for period t,

$$\delta_{jt} = G_{jt}/W_t = \{I(p^*_{jt-1} > 1)s_{jt} + [1 - I(p^*_{jt-1} > 1)] \max(s_{jt} - f_{jt}, 0)\}(1 - p^*_{jt-1}) \tag{14}$$

is the fraction of beginning-of-period wealth that is taxable as *realized* capital gains on stock j in period t, and p^*_{jt-1} is given by

$$p^*_{jt-1} = \begin{cases} \frac{[s_{jt-1}p^*_{jt-2} + \max(f_{jt-1} - s_{jt-1}, 0)]/(1+g_{jt})}{s_{jt-1} + \max(f_{jt-1} - s_{jt-1}, 0)}, & \text{if } \ p^*_{jt-2} < 1 \\ \frac{1}{1+g_{jt}}, & \text{if } \ p^*_{jt-2} \geq 1. \end{cases} \tag{15}$$

The linearity of the dynamic wealth equation and the assumption of constant relative risk averse preferences ensures that our two–asset model has the property that the consumption and portfolio decision rules, $\{c_t, b_t, f_{1t}, \cdots, f_{Jt}\}$, are independent of wealth, W_t. Furthermore, with the above transformation, the relevant state variables for the investor's problem become $x_t = \{s_{1t}, p^*_{1t-1}, \cdots, s_{Jt}, p^*_{Jt-1}\}$. Defining $v_t(x_t) = V_t(X_t)/[W_t/(1+i)^t]^{1-\gamma}$ to be the normalized value function and $w_{t+1} = W_{t+1}/[W_t(1+i)]$ to be the gross *real* growth rate in wealth from period t to period $t+1$, the investor's optimization problem can now be stated as follows:

$$v_t(x_t) = \max_{c_t, b_t, f_{1t}, \cdots, f_{Jt}} \left\{ \frac{e^{-\lambda_t} c_t^{1-\gamma}}{1-\gamma} + \frac{(1-e^{-\lambda_t})\beta(1-\beta^H)A_H^{1-\gamma}}{(1-\beta)(1-\gamma)} + e^{-\lambda_t}\beta E_t[v_{t+1}(x_{t+1})w_{t+1}^{1-\gamma}] \right\},$$
$$t = 0, \cdots, T-1, \tag{16}$$

s.t.

$$w_{t+1} = \frac{R_{t+1}}{1+i}(1 - \tau_g \sum_{j=1}^{J} \delta_{jt} - c_t), \qquad t = 0, \cdots, T-1, \tag{17}$$

$$f_{jt} \geq 0, \ j = 1, \cdots, J, \quad t = 0, \cdots, T-1, \tag{18}$$

where R_{t+1} is given by Eq. (12), δ_{jt} is given by Eq. (15), and p^*_{jt-1} is given by Eq. (16).

The above problem can be solved numerically using backward recursion. To do this, we discretize the lagged endogenous state variables, $x_t = \{s_{1t}, p^*_{1t-1}, \cdots, s_{Jt}, p^*_{Jt-1}\}$, into a grid. At the terminal date T, the investor's value function takes the value

$$v_T = \frac{\beta(1-\beta^H)A_H^{1-\gamma}}{(1-\beta)(1-\gamma)} \tag{19}$$

at all points in the state space. The value function at date T is then used to solve for the optimal decision rules and value function for all points on the grid at date $T-1$. Multilinear interpolation is used to calculate the value function for points in the state space that lie between the grid points. The procedure is repeated recursively for each time period until the solution for date $t=0$ is found.

Figure Legends

Figure 1. The top left panel shows the optimal overall equity proportion at age 99. The top right panel depicts the optimal company stock proportion at age 99. The bottom left panel presents the optimal stock index proportion at age 99. All three equity proportions are plotted against the company stock basis–price ratio and the stock index basis–price ratio. The initial company stock and the stock index holdings are set at $s_c = 0.9$ and $s_i = 0.1$, respectively. The capital gain returns for both the company stock and the stock index are set at 7%. The standard deviation of the company stock and the stock index are set at 40% and 20%, respectively. The correlation between the stock returns is set at 0.5.

Figure 2. The top left panel shows the optimal overall equity proportion at age 90. The top right panel depicts the optimal company stock proportion at age 90. The bottom left panel presents the optimal stock index proportion at age 90. All three equity proportions are plotted against the company stock basis–price ratio and the stock index basis–price ratio. The initial company stock and the stock index holdings are set at $s_c = 0.9$ and $s_i = 0.1$, respectively. The capital gain returns for both the company stock and the stock index are set at 7%. The standard deviation of the company stock and the stock index are set at 40% and 20%, respectively. The correlation between the stock returns is set at 0.5.

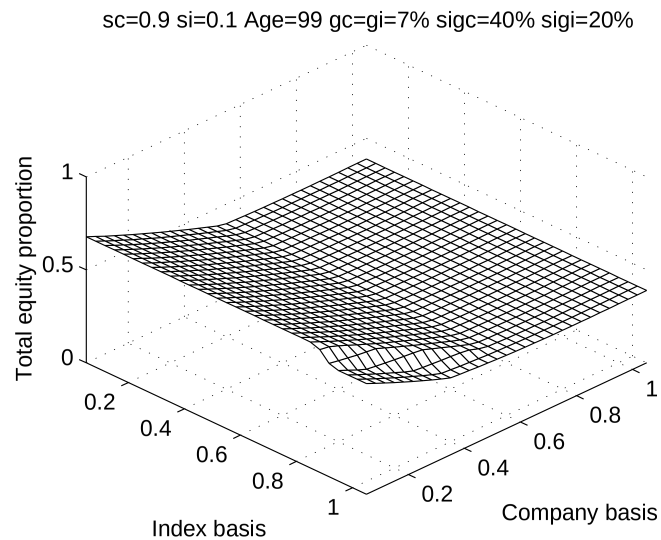
Figure 3. The top left panel shows the optimal overall equity proportion at age 90. The top right panel depicts the optimal company stock proportion at age 90. The bottom left panel presents the optimal stock index proportion at age 90. All three equity proportions are plotted against the company stock basis–price ratio and the stock index basis–price ratio. The initial company stock and the stock index holdings are set at $s_c = 0.5$ and $s_i = 0.5$, respectively. The capital gain returns for both the company stock and the stock index are set at 7%. The standard deviation of the company stock and the stock index are set at 40% and 20%, respectively. The correlation between the stock returns is set at 0.5.

Figure 4. The top left panel shows the optimal overall equity proportion at age 99. The top

right panel depicts the optimal company stock proportion at age 99. The bottom left panel presents the optimal stock index proportion at age 99. All three equity proportions are plotted against the company stock basis–price ratio and the stock index basis–price ratio. The initial company stock and the stock index holdings are set at $s_c = 0.5$ and $s_i = 0.5$, respectively. The capital gain returns for both the company stock and the stock index are set at 10%. The standard deviation of both the company stock and the stock index are set at 30%. The correlation between the stock returns is set at 0.5.

Figure 5. The top left panel shows the optimal overall equity proportion at age 40. The top right panel depicts the optimal company stock proportion at age 40. The bottom left panel presents the optimal stock index proportion at age 40. All three equity proportions are plotted against the company stock basis–price ratio and the stock index basis–price ratio. The initial company stock and the stock index holdings are set at $s_c = 0.5$ and $s_i = 0.5$, respectively. The capital gain returns for both the company stock and the stock index are set at 10%. The standard deviation of both the company stock and the stock index are set at 30%. The correlation between the stock returns is set at 0.5.

Figure 6. The no–trade regions for the company stock and the stock index at age 80 for the basis–price ratio of 0.05 (top left panel), the basis–price ratio of 0.5 (top right panel), and the basis–price ratio of 0.8 (bottom left panel). For the company stock, the steep solid line at the left depicts the lower bound of the no–trade region, while the line at the right shows the upper bound of the no–trade region. For the stock index, the flat solid line at the bottom represents the lower bound of no–trade region, while the line at the top shows the upper bound of no–trade region.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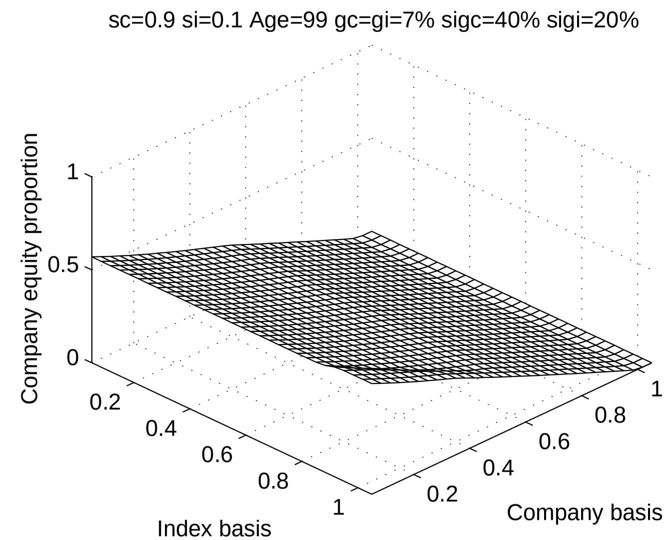
Company basis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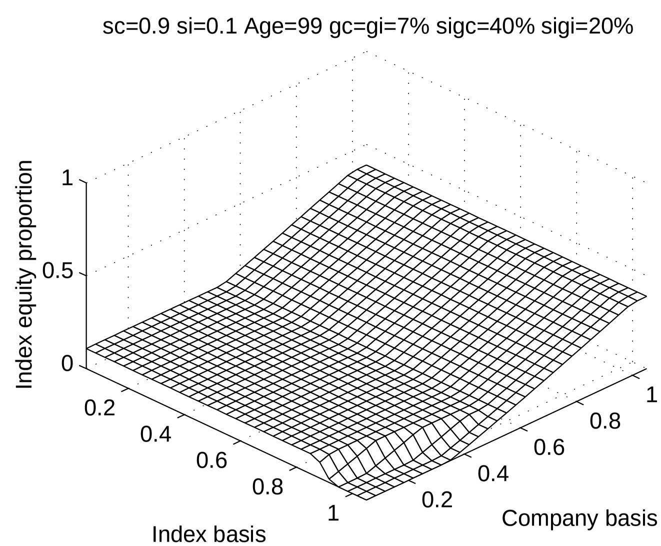
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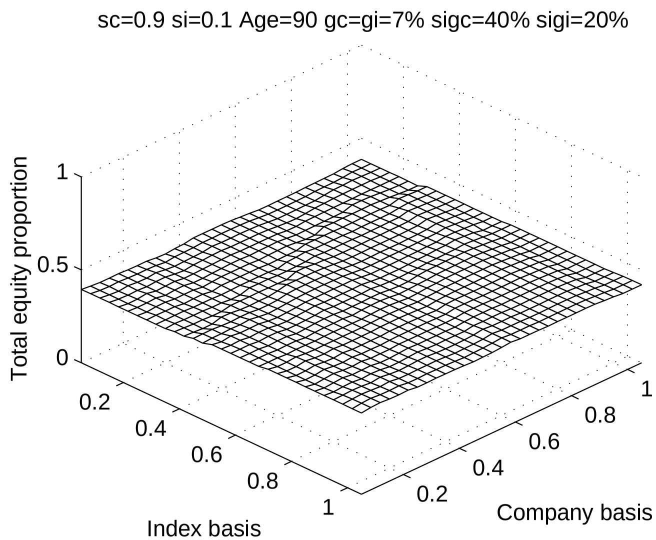
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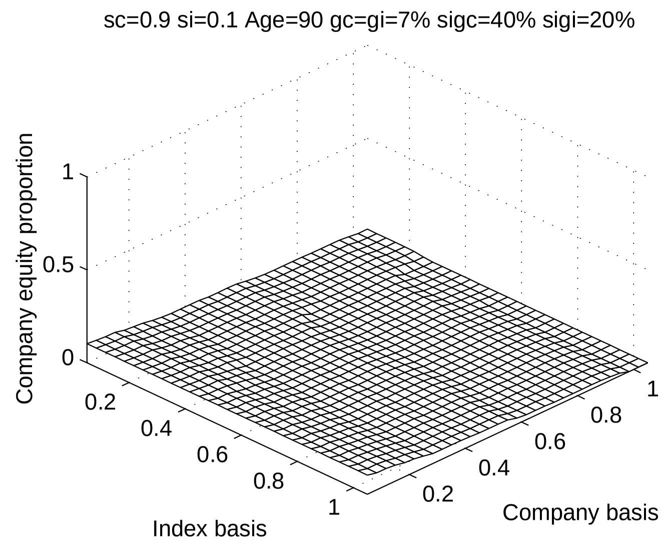
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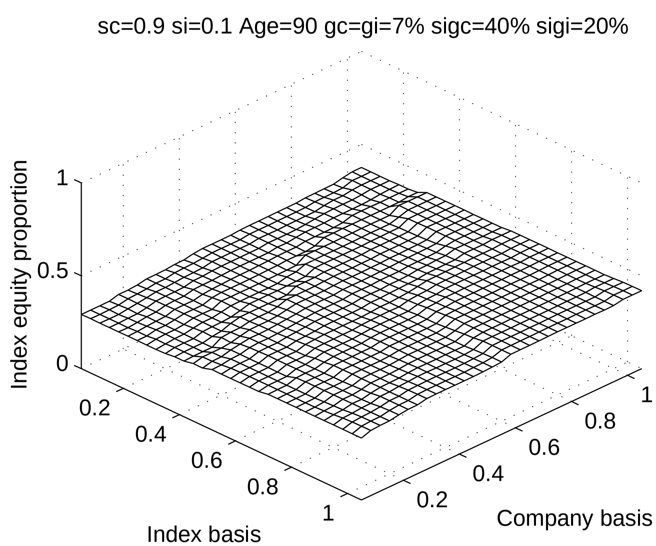
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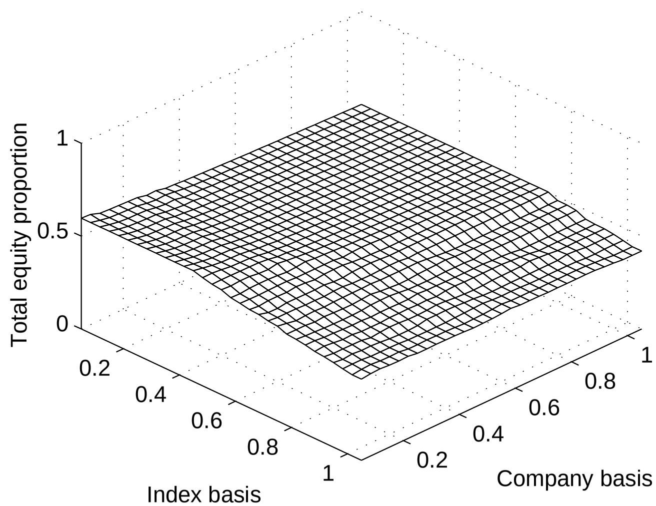

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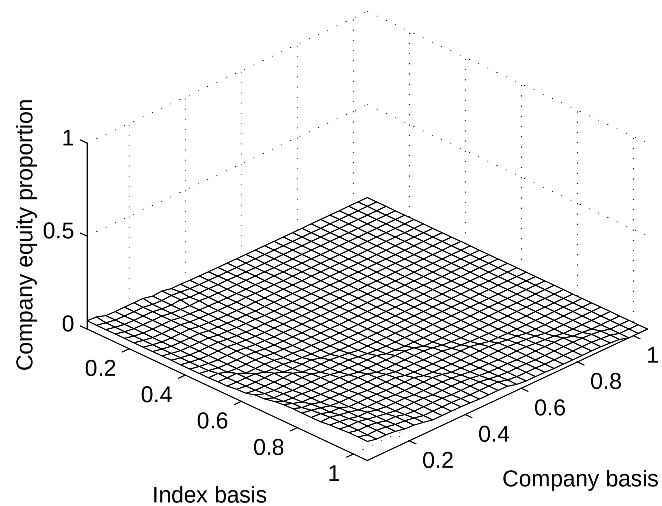

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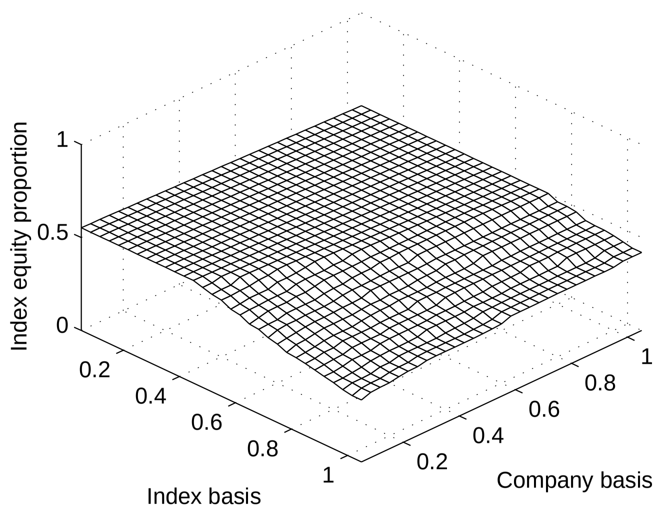

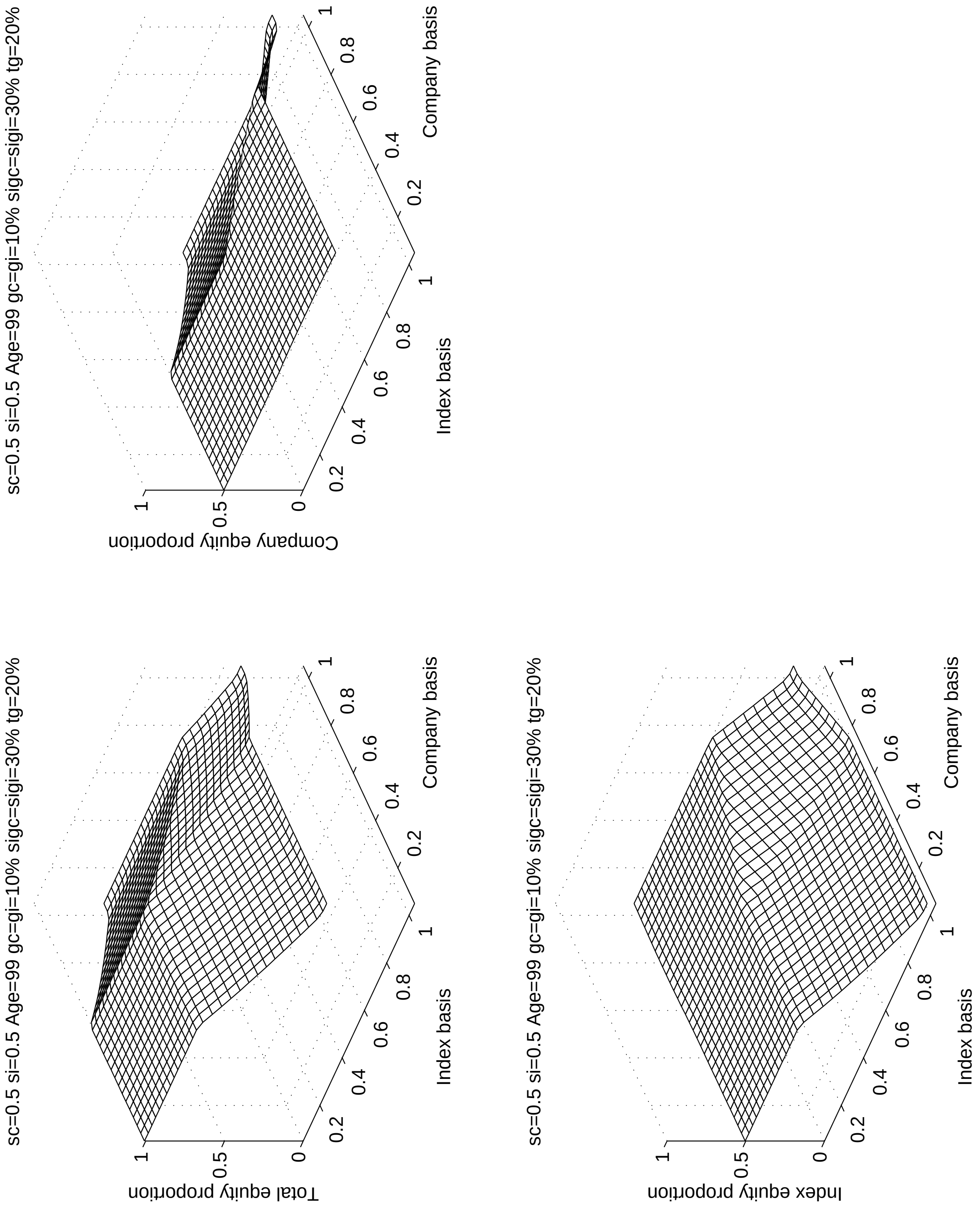
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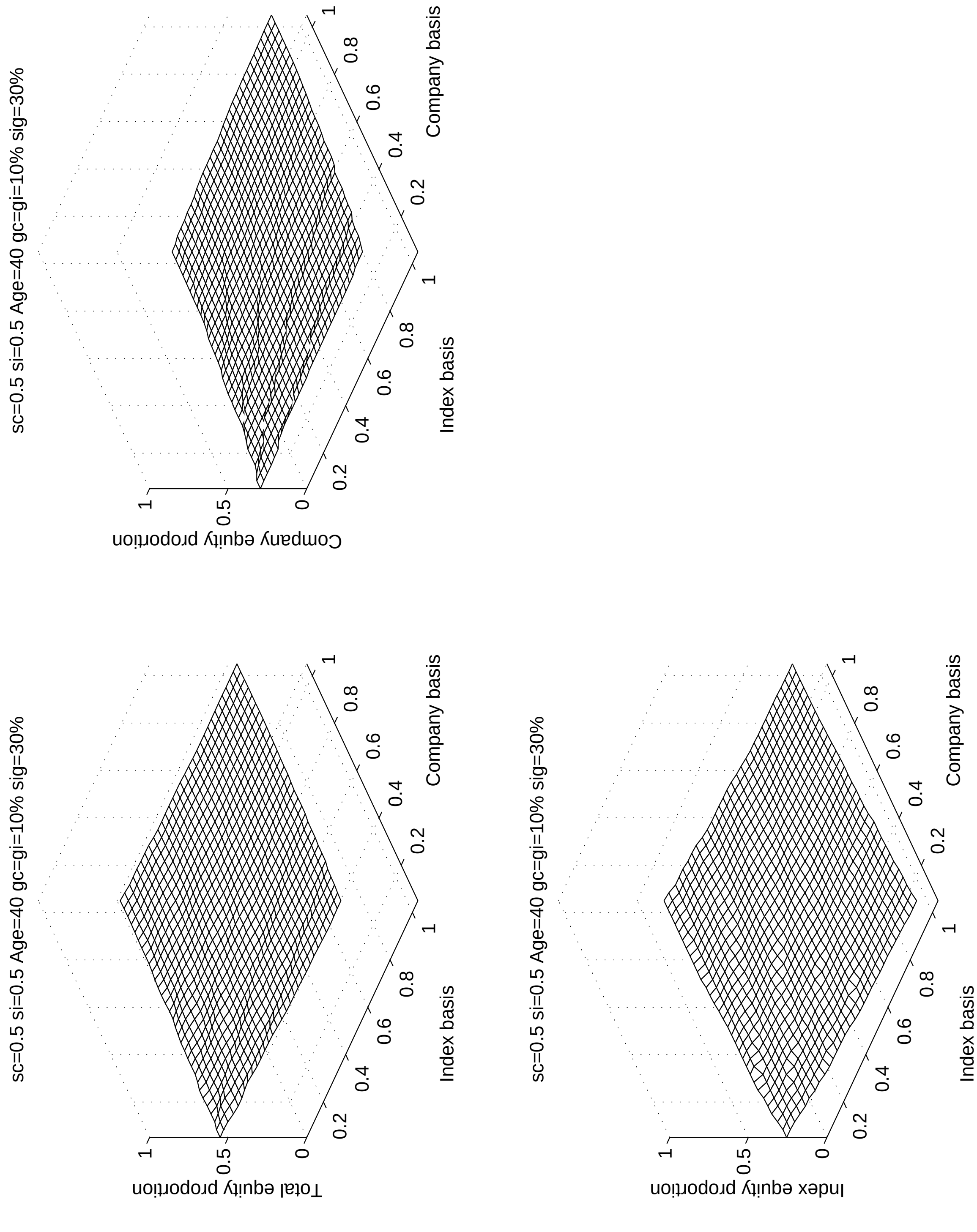

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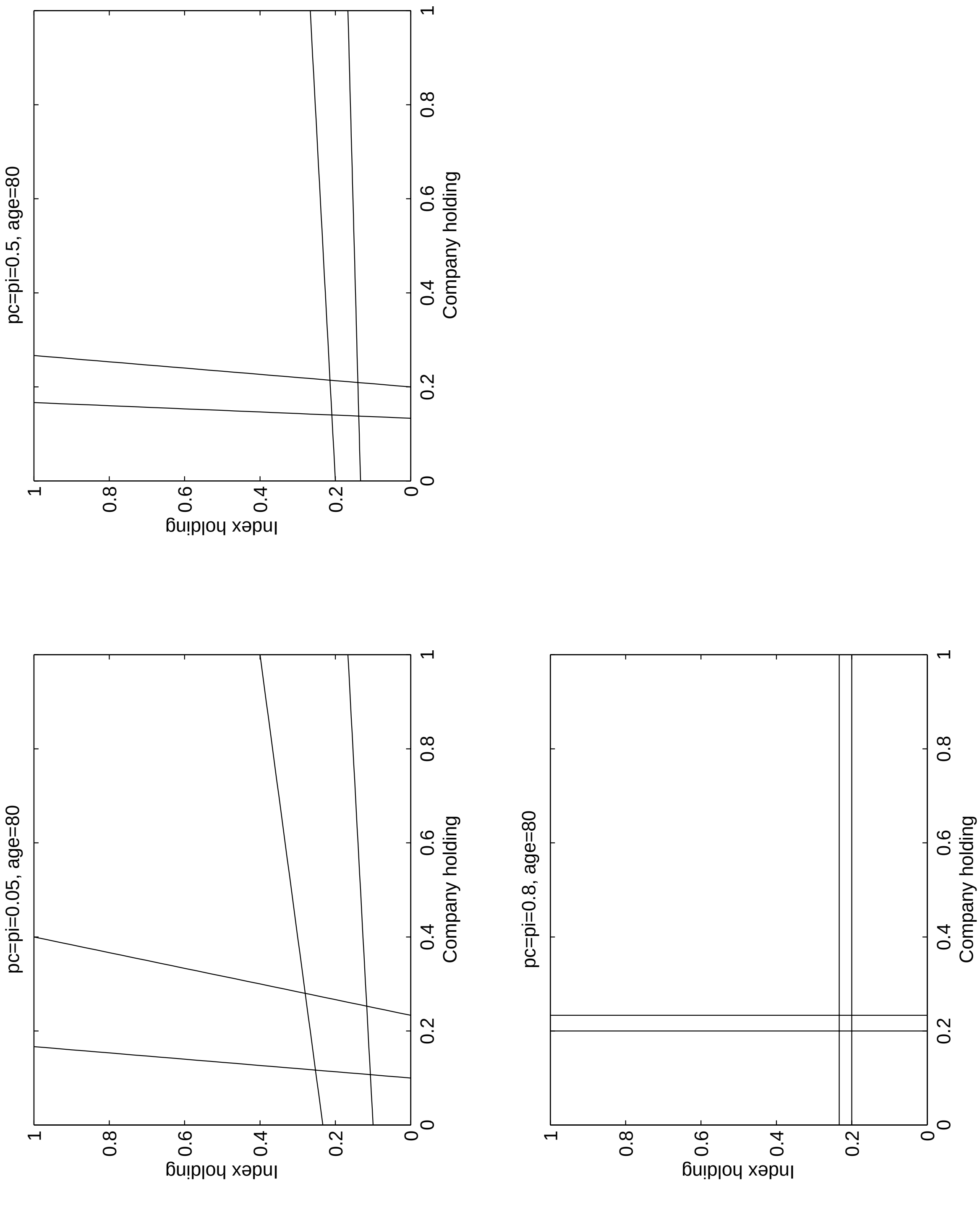
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